# APPENDIX - TREE METRICS AND LOG-CONCAVITY FOR MATROIDS

FEDERICO ARDILA–MANTILLA, SERGIO CRISTANCHO, GRAHAM DENHAM, CHRISTOPHER EUR, JUNE HUH, AND BOTONG WANG

## A. Equality cases

Here we aim to prove the statement of Proposition 2.3:

Let $T$ be a leaf-positive ultrametric tree of radius 1. Then $c_T = 1 - \frac{1}{n}$ if and only if $T$ is a star-metric.

Recall that for an ultrametric tree $T$:

(1) We say that $T$ is *leaf-positive* if every edge of $T$ adjacent to a leaf has positive weight. Note that this is equivalent to saying that for any vertex $i$ of $T$, $H(i) = 0$ if and only if $i$ is a leaf.

(2) We say that $T$ is a *star-metric* if, for any two leaves $i, j$ of $T$, the height of their lowest common ancestor $H(i \vee j)$ is either 0 or 1.

The leaf-positive property is relevant, since leaves on edges of weight zero can affect the value of $\hat{c}_T$ (see Example A.2). However, it suffices to settle the equality cases for leaf-positive trees, as the next result shows.

**Lemma A.1.** Let $T$ be an ultrametric tree of radius 1. Let $T'$ be the leaf-positive tree obtained from $T$ by contracting all length 0 pendants, then $\hat{c}_T = \hat{c}_{T'}$.

*Proof.* If $T$ is not leaf-positive, there exists a leaf $i$ of $T$ such that its parent $h$ satisfies $H(h) = 0$. There are two cases, either $i$ is isolated or $i$ has an adjacent leaf $j$.

If $i$ is isolated, let $T'$ be the ultrametric tree obtained from $T$ by contracting the edge between $i$ and $h$ (the pendant of $i$). Then the matrices $A_c$ of $T$ and $A'_c$ of $T'$ are equal, hence $\hat{c}_T = \hat{c}_{T'}$.

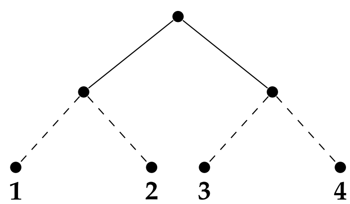
If $i$ has an adjacent leaf $j$, $d(i, j) = 0$ since $T$ is ultrametric, so $d(i, k) = d(j, k) = d(h, k)$ for every $k \neq i, j$ leaf. Let $T'$ be the ultrametric tree obtained from $T$ by contracting $i$ and $j$. Then the matrices $A(c)$ of $T$ and $A'(c)$ of $T'$ satisfy the following

$$A(c) = \begin{bmatrix} c & c & c - \frac{d(i,k)}{2} & \cdots \\ c & c & c - \frac{d(j,k)}{2} & \cdots \\ c - \frac{d(i,k)}{2} & c - \frac{d(j,k)}{2} & c & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & c & c - \frac{d(h,k)}{2} & \cdots \\ 0 & c - \frac{d(h,k)}{2} & c & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A'(c) \end{array}\right]$$

Hence $\hat{c}_T = \hat{c}_{T'}$. We obtain the result by iterating this process until $T'$ is leaf-positive. □

*Example* A.2. Let $T$ be the following binary tree with 4 leaves:




where the dashed edges have weight 0 and the solid edges have weight 1. $T$ is ultrametric but not leaf-positive, so it only defines a pseudometric on its set of leaves since $d(1,2) = 0$ and similarly for 3 and 4. The associated distance matrix is:

$$D = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$$

and one can compute that $\hat{c}_T = 1/2 < 3/4$. Note that $T$ can be contracted into the star tree $T'$ with two leaves, and in agreement with Proposition 2.3 we have $\hat{c}_T = 1/2$.

We now proceed to prove Proposition 2.3. for leaf-positive trees. In fact, we answer the more general question that arises naturally from Proposition 2.2:

If $T$ is a leaf-positive ultrametric tree of radius 1. Let $U$ be an upper subtree of $T$ and $M$ be the set of minimal elements of $U$. With the same notation as in Proposition 2.2, define the matrix $A^{T,U}(c)$ whose rows and columns indexed by $M$ with entries

$$(A^{T,U}(c))_{ij} = \begin{cases} c - H(i \vee j) & i \neq j \\ c - (1 - \frac{1}{n_i})H(i) & i = j. \end{cases}$$

Define $\hat{c}_T(U)$ as the smallest $c \geq 0$ that makes $A_c^U$ positive semi-definite. We know that $0 < \hat{c}_T(U) \leq 1 - \frac{1}{n}$, so when is $\hat{c}_T(U) = 1 - \frac{1}{n}$? We arrive at an answer by retracing the proof of Proposition 2.2, but first, let us prove the following lemma:

**Lemma A.3.** Let $T$ be a leaf-positive equidistant tree of height 1. Let $U$ be an upper subtree of $T$ and $M$ be the set of lowest elements of $U$. If $c = \hat{c}_T(U)$ then the matrix $A^{T,U}(c)$ is singular.

*Proof.* The matrix $A^{T,U}(0)$ is has at most one negative eigenvalue by [2, Thm. 3]. The matrix $A^{T,U}(c)$ is the rank 1 update of $A^{T,U}(0)$ by a multiple of the all-ones matrix $c\mathbf{1}_{n\times n}$, hence the eigenvalues of $A^{T,U}(c)$ increase continuously as $c$ increases. This is a direct consequence of the interlacing property for rank 1 updates [1, Thm. 1]. If $\lambda_1(c)$ is the smallest eigenvalue of $A^{T,U}(c)$ as a function of $c$, the previous statements say that $\lambda_1(c)$ is an increasing continuous function on $c$. We conclude that $\hat{c}_T(U)$ is the minimum $c$ for which $\lambda_1(c) = 0$, in particular, $A^{T,U}(c)$ is singular. □

**Proposition A.4** (Equality case of Proposition 2.2)**.** Let $T$ be a leaf-positive ultrametric tree of radius 1. Let $U$ be an upper binary subtree of $T$, $M$ be the set of lowest elements of $U$. For each $i \in M$, let $n_i$ be the number of leaves of $T$ below $i$, and let $n := \sum_{i\in M} n_i$. Then $\hat{c}_T(U) = 1 - \frac{1}{n}$ if and only if $U$ is star-metric and all the elements of $M$ have height 0 or 1.

*Remark* A.5. When $U$ equals the whole tree $T$, we have $\hat{c}_T = \hat{c}_T(U)$. Thus, Proposition 2.3 follows as a corollary.

*Proof.* Let us denote $\langle a \rangle = 1 - \frac{1}{a}$ for any $a \in \mathbb{Z}^+$. Like in Proposition 2.2 we may assume $T$ is binary.

First, we prove that $\hat{c}_T(U) = \langle n \rangle$ implies that $U$ is star-metric and all the elements of $M$ have height 0 or 1. We prove this statement for all pairs $(T, U)$ by induction on $|V(T)| + |M|$. The base case holds trivially; consider a larger pair $(T, U)$. At least one of the following statements is true:

(1) There is a leaf $i$ of $U$ with no siblings in $U$.

(2) There are sibling leaves $i, j$ of $U$ with a common parent $h = i \vee j$.

In the first case, by an argument identical to the one in proposition 2.2, we can obtain a tree $T'$ and an upper subtree $U'$ of $T'$ such that the lowest elements of $U'$ and $U$ are the same as well as their descendants in $T$ and $T'$ respectively. Furthermore $A^{T,U}(c) = A^{T',U'}(c)$ for all $c > 0$, hence $\hat{c}_{T'}(U') = \hat{c}_T(U) = \langle n \rangle$. Since the induction hypothesis applies to $(T', U')$, it follows that $U'$ is star-metric and all the elements of $M$ have height 0 or 1, this implies the same statement for $(T, U)$.

In the second case, by retracing the proof of Proposition 2.2, we arrive at the following equation

$$A^{T,U}(c) = B^{T,U}(c) + D,$$

where $D$ is the diagonal matrix $\operatorname{diag}(\langle n_i \rangle (H(h) - H(i)), \langle n_j \rangle (H(h) - H(j)))$ and $B_c^U$ is a matrix satisfying that

$$B^{T,U}(c) \sim \left[\begin{array}{c|c} (\frac{1}{n_i} + \frac{1}{n_j})H(h) & 0 \\ \hline 0 & A^{T',U'}(c) \end{array}\right]$$

where $U' = U \setminus \{i, j\}$ is the binary upper subtree of $T$ obtained by removing $i$ and $j$ from $U$. This proves the inequality $\hat{c}_T(U) \leq \hat{c}_T(U')$. Compounding this with our assumption that $\hat{c}_T(U) = \langle n \rangle$ and Proposition 2.2, we obtain that $\hat{c}_T(U') = \langle n \rangle$.

By the induction hypothesis, we know then that $U'$ is star-metric and all the elements of $M$ have height 0 or 1. Now, $h$ is a minimal element of $U'$, so it has to have height 0 or 1, but it cannot have height 0, as we have assumed $T$ is leaf-positive and $h$ is not a leaf as the ancestor of both $i$ and $j$, so $H(h) = 1$. This implies that $U$ is star-metric as $H(i \vee j) = 1$.

Now we are only left with proving that the heights of $i$ and $j$ must be 0 or 1. For this purpose, we use the explicit form of $A^{T,U}(c)$

$$(A^{T,U}(c))_{kl} = \begin{cases} c - 1 & k \neq l \\ c - \langle n_k \rangle H(k) & k = l. \end{cases}$$

For $c = \langle n \rangle$ we can rewrite the matrix as

$$A^{T,U}(\langle n \rangle) = (\langle n \rangle - 1)E(\operatorname{Id}_M + E^{-1}\mathbf{1}_{n \times n}),$$

where $\operatorname{Id}_M$ is the $|M| \times |M|$ identity matrix and $E$ is the diagonal matrix

$$E_{kl} = \begin{cases} 0 & k \neq l \\ \frac{1 - \langle n_k \rangle H(k)}{\langle n \rangle - 1} & k = l. \end{cases}.$$

The matrix $E$ is invertible since for every $k \in M$, $\langle n_k \rangle < 1$ and $H(k) \leq 1$. This way, we can compute the determinant of $A^{T,U}(\langle n \rangle)$ explicitly as

$$\det(A^{T,U}(\langle n \rangle)) = \left[\prod_{k \in M} (1 - \langle n_k \rangle H(k))\right] \left[1 - \frac{1}{n} \sum_{k \in M} \frac{n_k}{n_k(1 - H(k)) + H(k)}\right]. \tag{A.1}$$

By Lemma A.3 the previous determinant is zero, but this can only happen when

$$\sum_{k \in M} \frac{n_k}{n_k(1 - H(k)) + H(k)} = n.$$

Now, for each $k \in M$ $1 \leq n_k(1 - H(k)) + H(k) \leq n_k$, hence we have the following term by term inequality

$$n = \sum_{k \in M} \frac{n_k}{n_k(1 - H(k)) + H(k)} \leq \sum_{k \in M} n_k = n,$$

which in reality is an equality, so we must have term by term equality. This means that for every $k \in M$ $n_k(1 - H(k)) + H(k) = 1$, which holds only if for every $k \in M$ either $H(k) = 1$ or $H(k) = 0$ and $k$ is a leaf of $T$ since $T$ is leaf-positive. We conclude that $U$ is star-metric and the elements of $M$ have heights 0 or 1.

Now we prove the converse statement. Assume that $U$ is star-metric and the elements of $M$ have heights 0 or 1. Let $M_0 = \{k \in M \colon H(k) = 0\}$ and define $M_1 = \{k \in M \colon H(k) = 1\}$. If $|M_1| = 0$, then $n = |M|$ and

$$(A^{T,U}(c))_{kl} = \begin{cases} c-1 & k \neq l \\ c & k = l \end{cases} \tag{A.2}$$

for which we have already established $\hat{c}_T(U) = \langle n \rangle$. Let us prove that we can reduce to this case. If $|M_1| > 0$, initially:

$$(A^{T,U}(c))_{kl} = \begin{cases} c-1 & k \neq l \\ c & k = l \in M_0 \\ c - \langle n_k \rangle & k = l \in M_1 \end{cases}.$$

Let $i \in M_1$, note $n_i > 1$. Let $M' = M \sqcup \{i_2, \dots, i_{n_i}\}$ and let $B^{T,U}(c)$ be the $M' \times M'$ matrix:

$$B^{T,U}(c) = \left[\begin{array}{ccc|c} 2 - \langle 1 \rangle - \langle 1 \rangle & & & \\ & \ddots & & 0 \\ & & 2 - \langle n_i - 1 \rangle - \langle 1 \rangle & \\ \hline & 0 & & A^{T,U}(c) \end{array}\right],$$

where the first $n_i - 1$ rows and columns are indexed by $\{i_2, \dots, i_{n_i - 1}\}$. By reversing the row and column reduction in Proposition 2.2, working backwards, from $i_{n_i}$ towards $i_2$ on each of the new $n_i - 1$ rows and columns, the matrix $B^{T,U}(c)$ can be made similar to a matrix $C^{T,U}(c)$ with entries:

$$(C^{T,U}(c))_{kl} = \begin{cases} c-1 & k \neq l \\ c & k = l \in M_0 \cup \{i, i_2, \dots, i_{n_i}\} \\ c - \langle n_k \rangle & k = l \in M_1 \setminus \{i\} \end{cases},$$

and naturally $A^{T,U}(c)$ is positive semidefinite if and only if $C^{T,U}(c)$ is positive semidefinite. This process can be repeated for each $i \in M_1$ until one obtains an $n \times n$ matrix identical to (A.2), allowing us to conclude that $\hat{c}_T(U) = \langle n \rangle$. □

## References


[1] James R. Bunch, Christopher P. Nielsen, and Danny C. Sorensen. Rank-one modification of the symmetric eigenproblem. *Numerische Mathematik*, 31(1):31–48, March 1978. A

[2] R. L. Graham and H. O. Pollak. On the Addressing Problem for Loop Switching. *Bell System Technical Journal*, 50(8):2495–2519, October 1971. A